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LETTER TO THE EDITOR

On the 1D Ising spin glass system with random long-ranged interactions

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Abstract. We consider a one-dimensional Ising spin glass system with random interactions of the form $J_{ij}=\varepsilon_{ij}f(\gamma)\exp(-\gamma|i-j|)$, ε_{ij} being independent random variables. Generalizing Kac's approach for the one-dimensional gas we formulate the solution of the problem as that of obtaining the eigenvalues of a certain integral equation. In particular, as $\gamma\to 0$ we recover the results of the Sherrington–Kirkpatrick model and this new approach may give further information about the spin glass transition

Long ago it was pointed out (Ashkin and Lamb 1943) that long-range order comes from asymptotic degeneracy of the largest eigenvalue of the transfer matrix associated with the partition function of the system. This fact occurs in a host of models and Kac argued that this asymptotic degeneracy of the largest eigenvalue provides the mathematical mechanism for phase transitions (Kac 1968, Hemmer and Lebowitz 1976). The most notable case where this happens is in Onsager's solution of the two-dimensional Ising model in which the free energy is given in terms of the largest eigenvalue of the transfer matrix (Onsager 1944). In his now classic paper (Kac 1968), Kac showed that the Curie–Weiss and models based on weak long-range interactions despite its deficiences (dimension independent, interaction energy size dependent) are not all that different from those based on short-range interactions, as he succeeded in obtaining the Curie–Weiss results in terms of the largest eigenvalue of a certain linear operator associated with a linear chain in the limit of vanishing long-ranged interactions. In this work we consider an Ising spin glass chain with exponentially decaying interactions, $J_{ij}=f(\gamma)\exp(-\gamma|i-j|)$ whose solution is given in terms of an eigenvalue problem and which reduces to the SK model (Sherrington and Kirkpatrick 1975) in the limit $\gamma\to 0$. In this way besides obtaining the SK model from a 'bona fide' one-dimensional model we show that (as $\gamma\to 0$) the spin glass transition is associated with the existence of highly degenerate states. For one-dimensional models with other couplings see the review article of Binder and Young (1986).

We consider a one-dimensional model of N Ising spins $\sigma_i=\pm 1$ with Hamiltonian

$$H=\sum_{i<j} f(\gamma)\,\mathrm{e}^{-\gamma|i-j|}\varepsilon_{ij}\sigma_i\sigma_j \tag{1}$$

where the ε_{ij} are identically distributed independent Gaussian random variables with probability distribution

$$P(\varepsilon_{ij})=\frac{1}{\sqrt{2\pi}}\exp(-\varepsilon_{ij}^2/2) \tag{2}$$

and $f(\gamma)$ is some well behaved function with the limiting form $f(\gamma)=\sqrt{\gamma}$ as $\gamma\to 0$, to ensure non-trivial results in this limit. Note that γ^{-1} can be interpreted as the effective number of spins interacting with a given spin The quenched free energy is given by

$$\beta F=-\langle \ln \mathrm{Tr}(\mathrm{e}^{-\beta H})\rangle_{\mathrm{av}} \tag{3}$$

where $\langle g\rangle_{\mathrm{av}}$ is the average over the probability distributions in equation (2). Using the replica method (Edwards and Anderson 1975) the free energy per spin $f=F/N$ can be written as

$$-\beta F=\frac{1}{N}\left[\frac{\partial}{\partial n}\langle Z^n\rangle\bigg|_0\right] \tag{4}$$

with

$$\langle Z^n\rangle=\mathrm{Tr}_n \exp\left\{\frac{\beta^2\gamma}{2}\sum_{(\alpha,\beta)}\sum_{ij}\mathrm{e}^{-2\gamma|i-j|}\sigma_i^\alpha\sigma_i^\beta\sigma_j^\alpha\sigma_j^\beta\right\} \tag{5}$$

where $(\alpha\beta)$ means distinct replica pairs, Tr_n is the trace over nN spins and multiplicative constants have been dropped, and we take $f(\gamma)=\sqrt{\gamma}$. Now observing that the inverse of the matrix $A_{ij}=\mathrm{e}^{-2\gamma|i-j|}$ is tridiagonal (Kac 1968) we can rewrite

$$\langle Z^n\rangle=\mathrm{Tr}_n\int_{-\infty}^{\infty}\dots\int\prod_{(\alpha\beta)}\left\{\exp\left(\beta\sqrt{\gamma}\sum_i x_i^{\alpha\beta}\sigma_i^\alpha\sigma_i^\beta\right)\right.$$
$$\left.\times W(x_1^{\alpha\beta},x_2^{\alpha\beta},\dots,x_N^{\alpha\beta})\,\mathrm{d}x_1^{\alpha\beta}\,\mathrm{d}x_2^{\alpha\beta}\dots\mathrm{d}x_N^{\alpha\beta}\right\} \tag{6}$$

where

$$W(x_1,x_2,\dots,x_N)=\omega(x_1)P_\gamma(x_1|x_2)P_\gamma(x_2|x_3)\dots P_\gamma(x_{N-1}|x_N) \tag{7}$$

$$\omega(x)=\frac{1}{\sqrt{2\pi}}\exp(-x^2/2) \tag{8}$$

$$P_\gamma(x|y)=[2\pi(1-\mathrm{e}^{-4\gamma})]^{-1/2}\exp\left\{-\frac{(y-x\,\mathrm{e}^{-2\gamma})^2}{2(1-\mathrm{e}^{-4\gamma})}\right\}. \tag{9}$$

In equation (6) we have decoupled spins on different sites at the expenses of introducing a coupling between replicas and the random fields $x_i^{\alpha\beta}$. Following Kac's work (Kac 1968) we write equation (6) in terms of a symmetric multidimensional kernel as

$$\langle Z^n\rangle=\int_{-\infty}^{\infty}\dots\int h\{x_1^{\alpha\beta}\}\prod_{K=1}^{N-1}K(\{x_K^{\alpha\beta}\},\{x_{K+1}^{\alpha\beta}\})h\{x_N^{\alpha\beta}\}\prod_{(\alpha\beta)}\mathrm{d}x_1^{(\alpha\beta)}\mathrm{d}x_2^{(\alpha\beta)}\dots\mathrm{d}x_N^{(\alpha\beta)} \tag{10}$$

where

$$K(\{x^{\alpha\beta}\},\{y^{\alpha\beta}\})=g\{x^{\alpha\beta}\}\left[\prod_{(\alpha\beta)}\frac{\omega(x^{\alpha\beta})P_\gamma(x^{\alpha\beta}|y^{\alpha\beta})}{\sqrt{\omega(x^{\alpha\beta})\omega(y^{\alpha\beta})}}\right]g\{y^{\alpha\beta}\} \tag{11}$$

$$h\{x^{\alpha\beta}\}=g\{x^{\alpha\beta}\}\prod_{(\alpha\beta)}\sqrt{\omega(x^{\alpha\beta})} \tag{12}$$

$$g\{x^{\alpha\beta}\}=\left[\mathrm{Tr}_{\sigma^\alpha}\exp\left(\beta\sqrt{\gamma}\sum_{(\alpha\beta)}x^{\alpha\beta}\sigma^\alpha\sigma^\beta\right)\right]^{1/2} \tag{13}$$

where the trace is taken at a single site over all replica spins. Performing the integrals over the sets $\{dx_2^{\alpha\beta}\}, \{dx_3^{\alpha\beta}\}, \ldots, \{dx_{N-1}^{\alpha\beta}\}$, in (10) yields

$$\langle Z^n \rangle = \int_{-\infty}^{\infty} \ldots \int K^{(N-1)}(\{x_1^{\alpha\beta}\}, \{x_N^{\alpha\beta}\}) h\{x_1^{\alpha\beta}\} h\{x_N^{\alpha\beta}\} \prod_{(\alpha\beta)} dx_1^{\alpha\beta}\, dx_N^{\alpha\beta} \tag{14}$$

where $K^{(S)}(\{x^{\alpha\beta}\}, \{y^{\alpha\beta}\})$ is the S-iterate of the kernel. Since the kernel (11) is symmetric, positive definite, and of Hilbert–Schmidt type it has a discrete set of positive eigenvalues $\lambda_1 > \lambda_2 > \ldots$ and a complete set of eigenfunctions $\psi_1\{x^{\alpha\beta}\}, \psi_2\{x^{\alpha\beta}\}, \ldots$, and $K^{(S)}(\{x^{\alpha\beta}\}, \{y^{\alpha\beta}\})$ has the representation

$$K^{(S)}(\{x^{\alpha\beta}\}, \{y^{\alpha\beta}\}) = \sum_{j=1}^{\infty} \lambda_j^S \psi_j\{x^{\alpha\beta}\} \psi\{y^{\alpha\beta}\}. \tag{15}$$

Substituting (15) into (10) we obtain

$$\langle Z^n \rangle = \sum_{j=1}^{\infty} \lambda_j^{N-1} \left(\int_{-\infty}^{\infty} \ldots \int \psi_j\{x^{\alpha\beta}\} h\{x^{\alpha\beta}\} \prod_{(\alpha\beta)} dx^{\alpha\beta} \right)^2 \tag{16}$$

By substituting (16) in (4) and taking first the thermodynamic limit we can express f in terms of the largest eigenvalue of the kernel given by equation (11). In general within the matricial approach the free energy is solely expressed in terms of the largest eigenvalue while other quantities like correlation functions depend on all the eigenvalues. The eigenvalues λ_j are obtained from the integral equation

$$\int_{-\infty}^{\infty} \ldots \int K(\{x^{\alpha\beta}\}, \{y^{\alpha\beta}\}) \psi\{y^{\alpha\beta}\} \prod_{(\alpha\beta)} dy^{\alpha\beta} = \lambda \psi\{x^{\alpha\beta}\} \tag{17}$$

or, equivalently, from the equation

$$\exp(-\gamma V\{x^{\alpha\beta}\}) \exp\left[\sinh(2\gamma) \sum_{(\alpha\beta)} \frac{\partial^2}{\partial x_{\alpha\beta}^2} \right] \exp(-\gamma V\{x^{\alpha\beta}\}) \psi\{x^{\alpha\beta}\}$$
$$= \lambda\, e^{-\gamma n(n-1)/2}\, \psi\{x^{\alpha\beta}\} \tag{18}$$

where

$$\gamma V\{x^{\alpha\beta}\} = \tfrac{1}{2} \tanh(\gamma) \sum_{(\alpha\beta)} (x^{\alpha\beta})^2 - \log \operatorname{Tr} \exp\left(\beta\gamma \sum_{(\alpha\beta)} x^{\alpha\beta} \sigma^\alpha \sigma^\beta \right). \tag{19}$$

Equations (17), (18) and (19) are the generalization of Kac's results for the spin glass problem. Since for finite γ the uniform ($\varepsilon_{ij} = 1$) model does not have phase transition at finite T we expect the same to be true here. However, for small γ systematic perturbation expansion can be used to obtain the γ dependence (γ^{-1} is the range of the interaction) of the properties of the model.

The exponentials in equation (18) can be combined using the Baker–Hausdorff formula for operators A and B, and to order γ the eigenvalue problem equation (18) is equivalent to the multidimensional Schrödinger equation

$$\left\{ -\sum_{(\alpha\beta)} \frac{\partial^2}{\partial x^{\alpha\beta}} + \frac{1}{2} V\{x^{\alpha\beta}\} \right\} \psi\{x^{\alpha\beta}\} = \frac{1}{2} \left[E + \frac{n(n-1)}{2} \right] \psi\{\psi^{\alpha\beta}\} \tag{20}$$

where

$$V\{x^{\alpha\beta}\} = \frac{1}{2} \sum_{(\alpha\beta)} (x^{\alpha\beta})^2 - \frac{1}{\gamma} \log \operatorname{Tr} \exp\left(\beta\sqrt{\gamma} \sum_{(\alpha\beta)} x^{\alpha\beta} \sigma^\alpha \sigma^\beta \right) \tag{21}$$

and E is defined through

$$\lambda = e^{-E\gamma}. \tag{22}$$

Of course, the largest eigenvalue λ_1 is given by $\lambda_1 = \exp(-E_0\gamma)$ where E_0 is the smallest E obtained from equation (20). The minima of the potential $V\{x^{\alpha\beta}\}$ are obtained from the solutions of

$$\sqrt{\gamma}\, x^{\alpha\beta} = \frac{\beta\, \mathrm{Tr}\{\sigma^\alpha\sigma^\beta \exp(\beta\sqrt{\gamma}\, \Sigma_{(\alpha\beta)}\, x^{\alpha\beta}\sigma^\alpha\sigma^\beta)\}}{\mathrm{Tr}\exp(\beta\gamma\, \Sigma_{(\alpha\beta)}\, x^{\alpha\beta}\sigma^\alpha\sigma^\beta)} = \beta\langle\sigma^\alpha\sigma^\beta\rangle \tag{23}$$

or equivalently, they occur at $x^{\alpha\beta} = \beta q_{\alpha\beta}/\sqrt{\gamma}$ with $q_{\alpha\beta} = \langle\sigma^\alpha\sigma^\beta\rangle$ as defined above and

$$q_{\alpha\beta} = \frac{\mathrm{Tr}\{\sigma^\alpha\sigma^\beta \exp(\beta^2\, \Sigma_{(\alpha\beta)}\, q_{\alpha\beta}\sigma^\alpha\sigma^\beta)\}}{\mathrm{Tr}\exp(\beta^2\, \Sigma_{(\alpha\beta)}\, q_{\alpha\beta}\sigma^\alpha\sigma^\beta)}. \tag{24}$$

In the classical approximation, the minimum E is given by the minimum of $V\{x^{\alpha\beta}\}$ and

$$\gamma\left(E_0 + \frac{n(n-1)}{2}\right) = \min\left\{\frac{\beta^2}{2}\sum_{(\alpha\beta)^2} q^2_{\alpha\beta} - \log \mathrm{Tr}\exp\left(\beta^2 \sum_{(\alpha\beta)} q_{\alpha\beta}\sigma^\alpha\sigma^\beta\right)\right\}. \tag{25}$$

Substituting (25) in (22), and using equation (4) we obtain, in the limit $\gamma \to 0$, exactly the expression of the SK free energy per particle

$$\beta f = \lim_{n\to 0}\frac{1}{n}\min\left\{\frac{\beta^2}{2}\sum_{(\alpha\beta)} q^2_{\alpha\beta} - \log \mathrm{Tr}\exp\left(\beta^2 \sum_{(\alpha\beta)} q_{\alpha\beta}\sigma^\alpha\sigma^\beta\right)\right\}. \tag{26}$$

We have reformulated the mean-field theory of Ising spin glasses in terms of a chain with random long-ranged interactions in the limit of vanishing interactions. Of course, we have a much harder task in solving equation (20) than in the evaluation (26). On the other hand from (20) we may obtain much more information. It is well known that equation (26) leads to a paramagnetic phase above $T = T_c$ with all $q_{\alpha\beta} = 0$ and to a spin glass phase below T_c where $q_{\alpha\beta}$ becomes a function order parameter (as $n \to 0$). Thus, for finite γ, perturbation expansion around this solution can be carried out. Above T_c the potential $V\{x^{\alpha\beta}\}$ has a unique minimum at $q_{\alpha\beta} = 0$ and the solution to (20) gives the paramagnetic solution with eigenfunctions centred at the origin.

Below T_c, $q_{\alpha\beta} \neq 0$, and for integer n $V\{x^{\alpha\beta}\}$ develops 2^{n-1} minima (van Hemmen and Palmer 1979) at points $|q_{\alpha\beta}| = q$ differing only in the sign of their components and related by a sign symmetry (replica symmetric solution), which is due to be time-reversal invariance of the Hamiltonian equation (1). They form a cube centred with the unit cube in the $n(n-1)/2$-dimensional space of the variables $\{x^{\alpha\beta}\}$. Note that the present approach, by equation (25), does not require to sum over all the minima as happens in the steepest descent calculation which would lead to a divergence in the $n \to 0$ limit (van Hemmen and Palmer 1979).

In the ferromagnetic chain the splitting of the one-well potential into a double-well potential below T_c is the mathematical mechanism behind the phase transition, leading to asymptotic degeneracy of the two lowest eigenvalues as $\gamma \to 0$ because the two wells becomes too far apart, and to a phase transition in this limit (Kac 1968).

In the spin glass phase the analytic continuation $n \to 0$ to be made from a broken replica symmetry extremum. Or, more appropriately, from a broken replica permutation symmetry solution. All variables $\{x^{\alpha\beta}\}$ in the potential are equivalent, it has an obvious permutation symmetry among the replica labels, and broken replica symmetry solution amounts to consider the potential $V\{x^{\alpha\beta}\}$ as having many more wells than those just given by the sign symmetry for the replica symmetric solution. The solutions will be on the surface of a sphere inside the unit sphere and the zero modes of the SK model might be connected with this replica space permutation ('rotation') symmetry. One has thus the possibility of having a highly degenerate ground-state (in addition to the simple degeneracy of sign symmetry) which in turn means the existence of many phases. This scenario seems to be the case here and has been proven in another framework by many workers (Mézard *et al* 1987). We note that $|\psi|^2$ for the ground state is related to the degree of order of the system. For the pure ferromagnet (as $\gamma \to 0$) it is just a δ-function centred at the Curie–Weiss solution and we speculate that in the spin glass system it will be equal to Parisi's overlap probability distribution function $P(q)$. Probably metastable states and free energy barriers may be approached with the present framework by extending the Newman and Schulman arguments (Newman and Schulman 1977) about analytic continuation of the eigenvalues. It is tempting to consider the analytic continuation $n \to 0$ of equation (20). In this case the indices $\alpha, \beta = 1, 2, 3, \ldots, n$ in the variables $x^{\alpha\beta}$ must be reparametrized into two continuous indices z, $t \in [0, 1]$ (Jonsson 1982) and the eigenfunctions becomes a functional $\psi[x(z, t)]$ and so the order parameter is in general a function of two variables $q(z, t)$. For $\gamma \to 0$ equation (20) gives us back the free energy of the SK model. Otherwise the limit seems to be not trivial. To sum up, we have extended Kac's well known lattice gas approach to the spin glass problem.

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